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METRICS ON PRODUCTS OF SURFACES WITH NON-POSITIVE SECTIONAL CURVATURE

BY

OSCAR PERDOMO

Departamento de Matematicas, Facultad de Ciencias
Universidad del Valle, A. A. 25 360 Cali, Colombia
e-mail: osperdom@mafalda.univalle.edu.co

ABSTRACT

Let (S_i, g_i), $i = 1, 2$ be two compact riemannian surfaces isometrically embedded in euclidean spaces. In this paper we show that if $M = S_1 \times S_2$, then for any function $F\colon M \to \mathbf{R}$, the graph of F, i.e. the manifold $\{(x, F(x)) : x \in M\}$, does not have positive sectional curvature.

1. Introduction

Let M be a riemannian manifold and let T_pM denote the tangent vector space of M at p. The sectional curvature is the function that assigns the Gauss curvature at p of the surface built of geodesics starting at p and velocity vector in σ to any 2-dimensional space $\sigma \subset T_pM$. We say that the riemannian manifold M has positive sectional curvature if for every point $p \in M$ the sectional curvature $K(\sigma)$ of every 2-plane $\sigma \subset T_pM$ is positive. An example of such manifolds are the n-dimensional spheres of radius r, $S^n(r)$, with the metric induced by $\mathbf{R}^{n+1}$. In this case its sectional curvature is equal to $1/r^2$ for any 2-plane σ in T_pM. In general, the question of deciding if a given manifold admits a riemannian metric with positive sectional curvature is a difficult one; for example, the conjecture stating that no riemannian metric on $S^2 \times S^2$ has positive sectional curvature is known as Hopf's conjecture and remains unsolved. In this paper, we prove that a certain type of metric on a product of surfaces cannot have positive sectional curvature.

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2. Main theorem

In this section we state and prove the main theorem of this paper.

THEOREM 2.1: *Let $S_i \subset \mathbf{R}^{k_i}$, $i = 1, 2$ be two compact riemannian surfaces with the metric induced by the euclidean spaces. If $M = S_1 \times S_2$, in particular $M \subset \mathbf{R}^N$ with $N = k_1 + k_2$, then for any smooth function $F: M \to \mathbf{R}$, the manifold $\bar{M} = \{(x, F(x)) \in \mathbf{R}^{N+1} : x \in M\}$ with the metric induced by $\mathbf{R}^{N+1}$ does not have positive sectional curvature.*

Remark: A generalization of this theorem to functions with values in $\mathbf{R}^k$ will provide a proof of Hopf's conjecture.

Before proving this theorem, we fix some notation and prove some lemmas that will help us to relate the sectional curvature on M and $\bar{M}$. Let us denote by $\bar{\nabla}$ the connection on M with the metric induced by the embedding $\phi(x) = (x, F(x))$ of M in $\mathbf{R}^{N+1}$, and let us denote by ∇ the connection on M induced by $\mathbf{R}^N$. We will use the following notation.

1. If $m \in M$, we denote by $\bar{m}$ the point $(m, F(m))$. We denote by $\bar{M}$ the manifold $\phi(M) \subset \mathbf{R}^{N+1}$ with the metric induced by $\mathbf{R}^{N+1}$.
2. If a map $Y: M \to \mathbf{R}^N$ defines a tangent vector field on M, we denote by $\bar{Y}$ the vector field on $\bar{M}$ defined by $\bar{Y}(\bar{m}) = (Y(m), dF_m(Y(m)))$.
3. Given $p \in M$ and $v \in \mathbf{R}^N$, we denote by $v^T(p)$ the tangent orthogonal projection of v on T_pM. Given $w \in \mathbf{R}^{N+1}$, we denote by $w^{\bar{T}}(\bar{p})$ the tangent orthogonal projection of w on $T_{\bar{p}}\bar{M}$.

Since the manifold $\bar{M}$ is isometric to the manifold M with the metric induced by the embedding $\phi(p) = (p, F(p))$, then we also denote by $\bar{\nabla}$ the connection on $\bar{M}$. We will find the sectional curvature on $\bar{M}$ in terms of the sectional curvature of M and the derivatives of F. For any $m \in M \subset \mathbf{R}^N$, let $\{v_i : i = 1, \ldots, n\}$ be an orthonormal frame defined in an open neighborhood $U \subset M$ of m; note that each $v_i: U \to \mathbf{R}^N$ is a tangent vector field. Without loss of generality we may assume that the vector fields $\nabla_{v_i} v_j$ vanish at m for all i, j. We denote by ∇F the gradient vector of F as a function on M; since the frame of the vector field v_i's is orthonormal, then for any $p \in U$ we have that $\nabla F = \sum_{i=1}^n dF_p(v_i(p))v_i(p)$. Recall that the hessian of F is the symmetric 2-tensor given by $\mathrm{Hess}(F)(X, Y) = \langle \nabla_X(\nabla F), Y\rangle$ for any pair of tangent vector fields on M. For any $p \in U$, we define $F_i(p) = dF_p(v_i(p))$ and $F_{ij}(p) = (\mathrm{Hess}(F))_p(v_i(p), v_j(p))$. Before trying to find a relation between the sectional curvature of M and $\bar{M}$, we need to prove the following lemmas, LEMMA 2.1: (a) *The inverse of the matrix* $\{g_{ij}\}_{i,j=1}^n$ *defined by* $g_{ij} = \delta_{ij} + F_iF_j$ *is the matrix* $\{g^{ij}\}_{i,j=1}^n$ *defined by*

$$g^{ij} = \delta_{ij} - \frac{F_iF_j}{1+|\nabla F|^2}.$$

(b) *If* v *is a vector in* $\mathbf{R}^N$ *and* r *is a real number, then for any* $m \in M$ *we have that*

$$(v,r)^{\bar{T}} = (v^T, \langle v, \nabla F\rangle) + \frac{r - \langle v, \nabla F\rangle}{1+|\nabla F|^2}(\nabla F, |\nabla F|^2) = \overline{v^T + \frac{r - \langle v, \nabla F\rangle}{1+|\nabla F|^2}\nabla F}.$$

Proof of Lemma 2.1: A direct computation shows that $\sum_{j=1}^n g_{ij}g^{jk} = \delta_{ik}$, therefore (a) follows. Let us prove (b). For any $m \in M$ let us define $\{v_i : i = 1, \dots, n\}$ as above; then we have that the vectors

$$\{w_i = \bar{v}_i(\bar{m}) : i = 1, \dots, n\}$$

form a base for $T_{\bar{m}}\bar{M}$, therefore, there exist numbers $c_1, \dots, c_n$ such that

$$(v,r)^{\bar{T}}(\bar{m}) = \sum_{i=1}^n c_i w_i. \tag{1}$$

We have $\langle w_i, w_j\rangle = \langle (v_i, dF_p(v_i)), (v_j, dF_p(v_j))\rangle = \delta_{ij} + F_i(m)F_j(m) = g_{ij}$. If we multiply equation (1) by w_j, we obtain $\langle v, v_j\rangle + rF_j = \sum_{i=1}^n c_i g_{ij}$. Now if we multiply this equation by g^{jk} and sum from $j = 1$ to $j = n$, we obtain

$$\begin{aligned} c_k &= \sum_{j=1}^n (\langle v, v_j\rangle + rF_j)g^{jk} \\ &= \langle v, v_k\rangle + rF_k - \sum_{j=1}^n \left\{ \frac{\langle v, v_j\rangle F_jF_k}{1+|\nabla F|^2} + \frac{rF_jF_jF_k}{1+|\nabla F|^2} \right\} \\ &= \langle v, v_k\rangle + rF_k - \frac{\langle v, \nabla F\rangle F_k}{1+|\nabla F|^2} - \frac{r|\nabla F|^2F_k}{1+|\nabla F|^2} \\ &= \langle v, v_k\rangle + \frac{r - \langle v, \nabla F\rangle}{1+|\nabla F|^2}F_k. \end{aligned}$$

Plugging these values for c_k in equation (1) we obtain the first equality in (b).

For the other equality in (b), it is enough to check that if

$$u = v^T + \frac{r - \langle v, \nabla F\rangle}{1+|\nabla F|^2}\nabla F,$$

then

$$\overline{u} = (u, dF(u)) = (u, \langle u, \nabla F\rangle) = (v^T, \langle v, \nabla F\rangle) + \frac{r - \langle v, \nabla F\rangle}{1+|\nabla F|^2}(\nabla F, |\nabla F|^2).$$

This proves the lemma. ∎

LEMMA 2.2: *If $v, w\colon M \to \mathbf{R}^N$ are tangent vector fields on M, then the Levi Civita connection of the tangent vector fields $\bar{v}, \bar{w}\colon \bar{M} \to \mathbf{R}^{N+1}$ is given by*

$$\bar{\nabla}_{\bar{v}}\bar{w} = \overline{\nabla_v w + \frac{\operatorname{Hess} F(v,w)}{1+|\nabla F|^2}\nabla F}.$$

Proof of Lemma 2.2: Let $\alpha(t)$ be a smooth curve on $M \subset \mathbf{R}^N$ such that $\alpha(0) = m$ and $\alpha'(0) = v$ and define $\beta(t) = (\alpha(t), F(\alpha(t)))$. Since the riemannian metrics on M and $\bar{M}$ are those induced by the euclidean spaces, we have

$$\begin{aligned}(\bar{\nabla}_{\bar{v}}\bar{w})(\bar{m}) &= \Big(\frac{d}{dt}\bar{w}(\beta(t))\Big)^{\bar{T}}\Big|_{t=0} = \Big(\frac{d}{dt}w(\alpha(t))|_{t=0}, \frac{d}{dt}\langle\nabla F, w\rangle(\alpha(t))\Big)^{\bar{T}}\Big|_{t=0}\\ &= \Big(\frac{d}{dt}w(\alpha(t))|_{t=0}, \langle\nabla_v\nabla F, w\rangle + \langle\nabla F, \nabla_v w\rangle\Big)^{\bar{T}}.\end{aligned}$$

Using part (b) of Lemma 2.1 and the fact that $(\frac{d}{dt}w(\alpha(t))|_{t=0})^T = \nabla_v w$, we obtain

$$\begin{aligned}(\bar{\nabla}_{\bar{v}}\bar{w})(\bar{m}) &= \overline{\nabla_v w + \frac{\langle\nabla_v\nabla F, w\rangle + \langle\nabla F, \nabla_v w\rangle - \langle\nabla F, \nabla_v w\rangle}{1+|\nabla F|^2}\nabla F}\\ &= \overline{\nabla_v w + \frac{\operatorname{Hess} F(v,w)}{1+|\nabla F|^2}\nabla F}.\end{aligned}$$

This proves the lemma. ∎

Since $\operatorname{Hess}(F)$ is symmetric, we have as a corollary of Lemma 2.2 that if $[v, w]$ vanishes at $m \in M$, then $[\bar{v}, \bar{w}]$ also vanishes at $\bar{m} \in M$.

Let us denote the covariant derivative of the tensor $\operatorname{Hess}(F)$ by D^2dF, i.e. for any tangent vector fields X, Y and Z we have

$$D^2dF(X,Y;Z) = Z(\operatorname{Hess}(F)(X,Y)) - \operatorname{Hess}(F)(\nabla_Z X, Y) - \operatorname{Hess}(F)(X, \nabla_Z Y).$$

LEMMA 2.3: *Let R and $\bar{R}$ denote the curvature tensor of M and $\bar{M}$, respectively. For any tangent vector fields $X, Y, Z\colon M \to \mathbf{R}^N$ in M we have*

$$\begin{aligned}&\bar{R}(\bar{X},\bar{Y})\bar{Z}\\ &=\overline{R(X,Y)Z} + \frac{\operatorname{Hess}(F)(X,Z)}{1+|\nabla F|^2}\overline{\nabla_Y\nabla F} - \frac{\operatorname{Hess}(F)(Y,Z)}{1+|\nabla F|^2}\overline{\nabla_X\nabla F}\\ &\quad+\frac{D^2dF(X,Z;Y) - D^2dF(Y,Z;X)}{1+|\nabla F|^2}\overline{\nabla F}\\ &\quad+(|\nabla F|^2-1)\\ &\quad\times\frac{\operatorname{Hess}(F)(X,Z)\operatorname{Hess}(F)(Y,\nabla F) - \operatorname{Hess}(F)(Y,Z)\operatorname{Hess}(F)(X,\nabla F)}{(1+|\nabla F|^2)^2}\overline{\nabla F}.\end{aligned}\tag{2}$$

Proof of Lemma 2.3: Given $m \in M$, let $\{v_i : i = 1, \dots, n\}$ be an orthonormal frame defined as above. In order to prove the lemma, it is enough to prove equation (2) at the point m in the case $X = v_i$, $Y = v_j$ and $Z = v_k$. Using the definition of curvature tensor and Lemma 2.2, we have

$$\begin{aligned}
&\bar{R}(\bar{v_i}, \bar{v_j})\bar{v_k} \\
&= \bar{\nabla}_{\bar{v_j}} \bar{\nabla}_{\bar{v_i}} \bar{v_k} - \bar{\nabla}_{\bar{v_i}} \bar{\nabla}_{\bar{v_j}} \bar{v_k} \\
&= \bar{\nabla}_{\bar{v_j}} \Big(\overline{\nabla_{v_i} v_k} + \frac{F_{ik}}{1+|\nabla F|^2} \overline{\nabla F} \Big) - \bar{\nabla}_{\bar{v_i}} \Big(\overline{\nabla_{v_j} v_k} + \frac{F_{jk}}{1+|\nabla F|^2} \overline{\nabla F} \Big) \\
&= \overline{\nabla_{v_j} \nabla_{v_i} v_k} + \overline{\nabla_{v_j} \Big(\frac{F_{ik}}{1+|\nabla F|^2} \nabla F \Big)} + \mathrm{Hess}(F) \Big(v_j, \frac{F_{ik}}{1+|\nabla F|^2} \nabla F \Big) \overline{\nabla F} \\
&\quad - \overline{\nabla_{v_i} \nabla_{v_j} v_k} - \overline{\nabla_{v_i} \Big(\frac{F_{jk}}{1+|\nabla F|^2} \nabla F \Big)} - \mathrm{Hess}(F) \Big(v_i, \frac{F_{jk}}{1+|\nabla F|^2} \nabla F \Big) \overline{\nabla F} \\
&= \overline{R(v_i, v_j) v_k} + \frac{F_{ik}}{1+|\nabla F|^2} \overline{\nabla_{v_j} \nabla F} - \frac{F_{jk}}{1+|\nabla F|^2} \overline{\nabla_{v_i} \nabla F} \\
&\quad + \Big\{ \frac{F_{ik}\, \mathrm{Hess}(F)(v_j, \nabla F) - F_{jk}\, \mathrm{Hess}(F)(v_i, \nabla F)}{1+|\nabla F|^2} \Big\} \overline{\nabla F} \\
&\quad + \Big\{ v_j \Big(\frac{F_{ik}}{1+|\nabla F|^2} \Big) - v_i \Big(\frac{F_{jk}}{1+|\nabla F|^2} \Big) \Big\} \overline{\nabla F}.
\end{aligned}$$

Using that $\nabla_{v_j} v_i(m)$ vanish for any $i, j \in \{1, \dots, n\}$, we can prove that

$$v_l(F_{rs}) = D^2 dF(v_r, v_s; v_l)$$

and

$$v_l((1+|\nabla F|^2)^{-1}) = -2(1+|\nabla F|^2)^{-2}\, \mathrm{Hess}(F)(v_l, \nabla F)$$

These equalities together with the expression for $\bar{R}(\bar{v_i}, \bar{v_j})\bar{v_k}$ that we obtained above give us the proof of the lemma.

Proof of Theorem 2.1: By Morse theory [B], there exists a closed geodesic $\gamma_1 \subset S_1$ and a closed geodesic $\gamma_2 \subset S_2$. Let us define

$$T = \{(x, y) \in M : x \in \gamma_1 \text{ and } y \in \gamma_2\}.$$

Let $h\colon T \to \mathbf{R}$ be the function defined by $h(m) = F(m)$ for all $m \in T$. For every $z \in S_i$, let $\psi_i\colon \gamma_i \to \mathbf{R}^{k_i}$ be unit tangent vector fields, $i = 1, 2$. Let $\phi_i\colon T \to \mathbf{R}^N$ be the tangent vector field defined by $\phi_1(x, y) = (\psi_1(x), 0, \dots, 0)$ and $\phi_2(x, y) = (0, \dots, 0, \psi_2(y))$. We will first prove the theorem in the case that h is a Morse function on T. Let $(x_0, y_0) \in T$ be a critical point of h which is a saddle point, i.e. the determinant of $\mathrm{Hess}(h)$ at (x_0, y_0) is negative; this saddle point exists because T is topologically a torus. For any $(x, y) \in T$ we can write $\nabla F(x,y) = \nabla h + (\nabla F)^{\perp}$, where $(\nabla F)^{\perp}$ is perpendicular to any vector in $T_{(x,y)}T$. It is not difficult to show that T is a totally geodesic submanifold of M [D], therefore we have

$$\begin{aligned}\mathrm{Hess}(F)(\phi_i,\phi_j) &= \langle \nabla_{\phi_i}\nabla F, \phi_j\rangle \\ &= \langle \nabla_{\phi_i}\nabla h, \phi_j\rangle + \langle \nabla_{\phi_i}(\nabla F)^{\perp}, \phi_j\rangle \\ &= \langle D_{\phi_i}\nabla h, \phi_j\rangle - \langle (\nabla F)^{\perp}, \nabla_{\phi_i}\phi_j\rangle \\ &= \mathrm{Hess}(h)(\phi_i,\phi_j).\end{aligned}$$

In the last equality above we have used the fact that since T is totally geodesic, then $\nabla_{\phi_i}\phi_j(x,y) \in T_{(x,y)}T$ for every $(x,y) \in T$; the D in the expression above denotes the Levi Civita connection on T.

Recall that $\langle \bar{X}, \bar{Y}\rangle = \langle X, Y\rangle + dF(X)dF(Y)$. We will prove the theorem by showing that $\langle \bar{R}(\bar{\phi_1},\bar{\phi_2})\bar{\phi_1},\bar{\phi_2}\rangle$ at $(x_0, y_0, F(x_0,y_0)$ is negative. Since (x_0,y_0) is a critical point in M, then $dh_{(x_0,y_0)}(\phi_i) = dF_{(x_0,y_0)}(\phi_i) = \langle \nabla F, \phi_j\rangle(x_0,y_0)$ vanishes. Using Lemma 2.3 and the fact that the sectional curvature of the plane spanned by $\{\phi_1,\phi_2\}$ is zero because M has the product metric [D], we obtain at the point (x_0,y_0) that

$$\begin{aligned}\langle \bar{R}(\bar{\phi_1},\bar{\phi_2})\bar{\phi_1},\bar{\phi_2}\rangle =& \langle R(\phi_1,\phi_2)\phi_1,\phi_2\rangle + \frac{\mathrm{Hess}(F)(\phi_1,\phi_1)}{1+|\nabla F|^2}\langle \nabla_{\phi_2}\nabla F, \phi_2\rangle \\ &- \frac{\mathrm{Hess}(F)(\phi_2,\phi_1)}{1+|\nabla F|^2}\langle \nabla_{\phi_1}\nabla F,\phi_2\rangle \\ =& \frac{\mathrm{Hess}(h)(\phi_1,\phi_1)\,\mathrm{Hess}(h)(\phi_2,\phi_2)}{1+|\nabla F|^2} \\ &- \frac{\mathrm{Hess}(h)(\phi_2,\phi_1)\,\mathrm{Hess}(h)(\phi_1,\phi_2)}{1+|\nabla F|^2} \\ <& 0.\end{aligned}$$

This inequality completes the proof in the case that the function $h\colon T \to \mathbf{R}$ is a Morse function.

For the general case we will proceed by contraction. Let us assume that M has positive sectional curvature. Since M is compact, then there exists $\epsilon > 0$ such that for any σ in T_pM, we have $K(\sigma) > \epsilon$. Let T be defined as above. Since M is compact, we can use the formula in Lemma 2.3 for the curvature tensor in order to find a positive δ such that if $\bar{F}\colon M \to \mathbf{R}$ is chosen such that the difference between $\bar{F}$ and its derivatives up to third order with F and its derivatives up to third order respectively is less than δ, then the difference between the sectional curvature of the graph of F and the graph of $\bar{F}$ is less than ϵ. By Morse Theory [M] we can choose this $\bar{F}$ such that the function $\bar{F}$ restricted to T is a Morse function. Using the case we already considered, we have that the graph of $\bar{F}$ has a plane with negative sectional curvature. This contradicts the fact that the difference between the sectional curvature of the graph of F and the graph of $\bar{F}$ is less than ϵ. ∎

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References

[B] R. Bott, *Lectures on Morse Theory, old and new*, Bulletin of the American Mathematical Society **7** (1982), 331–358.

[D] M. Do Carmo, *Riemannian Geometry*, Birkhäuser, Basel, 1992.

[M] J. Milnor, *Morse Theory*, Princeton University Press, 1963.